Rates of Convergence to Self-Similar Solutions of Burgers' Equation

By Joel C. Miller and Andrew J. Bernoff

We study the large-time behavior of solutions to Burgers' equation with localized initial conditions. Previous studies have demonstrated that these solutions converge to a self-similar asymptotic solution $\Theta(x, t)$ with an error whose L_p norm is of order $t^{-1+1/2p}$. Noting that the temporal and spatial translational invariance of the underlying equations leads to a two-parameter family of self-similar solutions $\Theta(x - x_*, t + t_*)$, we demonstrate that the optimal choice of x_* and t_* reduces the L_p error to the order of $t^{-2+1/2p}$.

1. Introduction

We consider the large-time behavior of solutions to Burgers' equation $u_t + cuu_x = \nu u_{xx}$ on the line $-\infty < x < \infty$ which arises in applications modeling traffic flow, fluid flow in conditions such as magneto-hydrodynamics, atmospheric behavior, and many other physical systems (cf. [1]). Scaling space and time, the equation becomes

$$u_t + uu_x = u_{xx}, \tag{1}$$

$$u(x, 0) = f(x). \tag{2}$$

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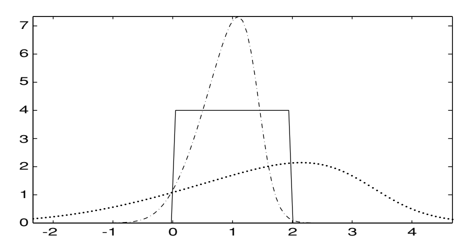
This is one of the simplest examples of a nonlinear partial differential equation, and thus it is useful as an example for studying their behavior.

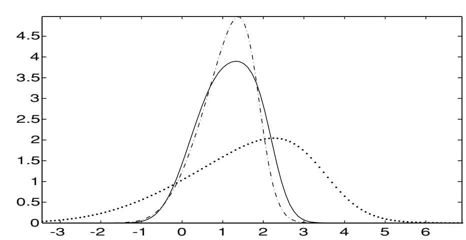
Several authors have investigated the large-time self-similar behavior of Burgers' equation. A self-similar solution is found in Whitham [1]. Grundy [2] studied the asymptotics of convergence to similarity solutions in generalized convective diffusion equations, of which Burgers' equation is a special case. Some more recent work by Chern [3, 4], Chern and Liu [5], Escobedo and Zuazua [6], and Zuazua [7] gives rates of convergence to self-similar solutions. Chern and Liu found a self-similar asymptotic solution which differs from the true solution by an error whose L_p norm is $\mathcal{O}(t^{-1+1/2p})$. Zuazua and Escobedo studied a generalization of Equation (1) in higher dimensions and found a self-similar asymptotic solution with a similar error.

Previous work has focused on finding a self-similar approximation $\Theta(x, t)$ by matching the mass of Θ with that of the true solution. We improve on these results by searching for a self-similar approximation of the form $\theta(x, t) = \Theta(x - x_*, t + t_*)$ and finding optimal values for x_* and t_*, corresponding to fitting the location and the width of the self-similar solution. This allows us to improve on the result of Chern and Liu by a factor of $1/t$.

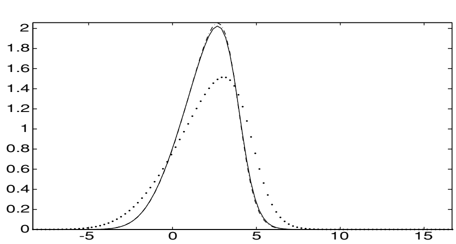
This improvement is demonstrated in Figure 1 which shows the true solution to a tophat initial condition as well as our asymptotic self-similar approximation



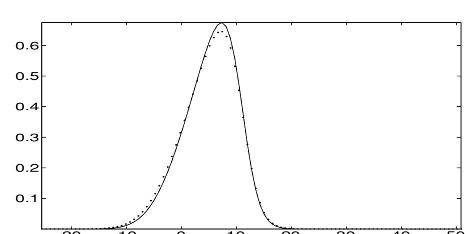
(a) $t = 0$



(b) $t = 0.1$



(c) $t = 1$



(d) $t = 10$

Figure 1. Solutions of Burgers' equation with a tophat initial condition (solid), the self-similar approximation due to Chern and Liu (dotted), and our self-similar approximation (dash-dot). The axes are scaled so that our approximation appears unchanging. At $t = 1$, our approximation is almost indistinguishable from the true solution.

and the previous approximation from the literature which had been shown to be "optimal" (in the absence of shifts in x and t).

To state our main result, we first define

$$M_n(f) = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

to be the nth moment of f. Further, $\rho(f)$ will be defined by

$$\rho(f) = \int_{-\infty}^{\infty} |x^3 f(x)|\, dx.$$

In addition, we will assume that the initial condition, $f(x)$, is piecewise continuous, which guarantees that $u(x, t)$ is smooth for all $t > 0$.

We obtain the following result:

THEOREM 1. *Assume that* $f(x) \geq 0$ *for all* x, *and let* u *be the solution of Equation* (1) *with initial condition* (2). *If* $\rho(f) < \infty$ *and* $0 < M_0(f) < \infty$ *then there exists a self-similar solution to Equation* (1), $\theta(x, t)$, *such that*

$$\|\theta(x, t) - u(x, t)\|_p = \mathcal{O}(t^{-2+1/2p}),$$

where $\|\cdot\|_p$ *denotes the* L_p *norm and* $1 \leq p \leq \infty$.

The assumption of non-negative f simplifies the analysis; we believe it can be relaxed significantly.

To achieve this result, we will first transform the problem from Burgers' equation to the heat equation using the Cole–Hopf transformation. Following that, we present and prove the corresponding theorem for the heat equation. Finally, we return to Burgers' equation by inverting the transformation and show the convergence by means of an example. We also include a useful asymptotic expansion for the heat equation, which elucidates the origins of the error estimates in both our work and that of Chern and Liu.

2. Transformation to the heat equation

We are interested in the large-time asymptotic behavior of solutions to Burgers' equation (1),

$$u_t + uu_x = u_{xx},$$

with a positive initial condition $f(x) \geq 0$, satisfying $\rho(f) < \infty$ and $0 < M_0(f) < \infty$.

To begin our analysis we linearize Burgers' equation using the Cole–Hopf transformation [1],

$$u = -2\phi_x/\phi, \qquad \phi(x,t) = \exp\left[-\frac{1}{2}\int_{-\infty}^{x} u(s,t)\,ds\right], \tag{3}$$

which reduces the initial value problem for Burgers' equation, to an initial-value problem for the heat equation,

$$\phi_t = \phi_{xx}, \qquad \phi(x,0) = \exp\left[-\frac{1}{2}\int_{-\infty}^{x} f(s)\,ds\right].$$

Note that the integral in the exponential increases monotonically from 0 to $M_0(f)$ as x increases. Consequently, using the maximum principle of the heat equation [8] we obtain that for $t > 0$

$$0 < \exp[-M_0(f)/2] \le \phi(x,t) \le 1 < \infty. \tag{4}$$

We conclude $\phi(x,t)$ is bounded but need not decay as $x \to \pm\infty$.

If we consider $\psi = -\phi_x$, ψ will solve the heat equation with initial conditions that decay as $x \to \pm\infty$,

$$\psi_t = \psi_{xx}, \qquad \psi(x,0) = \frac{1}{2} f(x) \exp\left[-\frac{1}{2}\int_{-\infty}^{x} f(s)\,ds\right] \equiv h(x). \tag{5}$$

Because the exponential term is at most 1 and at least $e^{-M_0(f)/2}$, we know that $h(x)$ is bounded above by $f(x)/2$ and from below by $f(x)e^{-M_0(f)/2}/2$. Thus $\rho(h)$ and the moments of h can be bounded in terms of the corresponding values for f. In particular, the existence of $\rho(h)$ and $M_0(h)$ is guaranteed by the existence of $\rho(f)$ and $M_0(f)$. Further, $M_0(f) > 0$ implies that $M_0(h) > 0$.

For later reference, we recover ϕ from ψ,

$$\phi(x,t) = 1 - \int_{-\infty}^{x} \psi(s,t)\,ds \tag{6}$$

and u from ψ and ϕ by modifying Equation (3),

$$u = 2\psi/\phi. \tag{7}$$

3. Analysis of the heat equation

We have recast the original Burgers' equation problem in terms of the heat equation

$$\psi_t = \psi_{xx}, \qquad \psi(x,0) = h(x),$$

where $h(x)$ satisfies $0 \le h(x)$, $0 < M_0(h) < \infty$, and $\rho(h) < \infty$.

We want to find a self-similar asymptotic solution of the heat equation to which ψ converges at large t. This problem was considered by Kleinstein and

Ting [9] (see also Witelski and Bernoff [10]) who showed that that at large time the solution approaches a Gaussian, with an error of order t^{-2}.

The Gaussian

$$G(x,t) = C\frac{\exp\left[-(x-x_*)^2/4(t+t_*)\right]}{\sqrt{4\pi(t+t_*)}} \tag{8}$$

is a three-parameter family of self-similar solutions to the heat equation, as long as $t_* > 0$. Let $g(x) = G(x,0)$ be the initial condition associated with this self-similar solution. By choosing C, x_*, and t_*, we can make the moments $M_0(g)$, $M_1(g)$, and $M_2(g)$ take on any values. Our goal is to choose the parameters such that the first three moments of g are equal to those of h. Given that, we will show that this maximizes the rate at which ψ converges to $G(x,t)$.

Choosing

$$\begin{aligned} C &= M_0(h)\\ x_* &= M_1(h)/M_0(h)\\ t_* &= \left[M_2(h)M_0(h) - M_1(h)^2\right]/2M_0(h)^2 \end{aligned}$$

matches the zeroth, first, and second moments of g with those of h [10]. An equivalent expression for t_* is

$$t_* = \frac{1}{2M_0}\int_{-\infty}^{\infty}(x-x_*)^2 h(x)\,dx,$$

so $t_* > 0$.

By superposition, the difference $\psi - G$ solves the heat equation. We define this to be

$$E \equiv \psi - G. \tag{9}$$

We now develop bounds on the rate of decay of the error term $\|E\|_p$.

THEOREM 2. *Assume that* $h(x) \geq 0$. *If* $\rho(h) < \infty$ *and* $0 < M_0(h) < \infty$, *then*

$$\|E(x,t)\|_p = \mathcal{O}(t^{-2+1/2p})$$

$$\left\|\int_{-\infty}^{x} E(s,t)\,ds\right\|_\infty = \mathcal{O}(t^{-3/2}).$$

Note that the bound on the integral of E is L_∞, not L_p. Although a similar L_p bound may be derived using the same techniques, it is unimportant here.

Before we can prove this theorem, we need to show that g exists given the conditions on h. That is, we need to show that C, x_*, and t_* are real numbers as defined above. To accomplish this, we show the existence of $M_1(h)$ and

$M_2(h)$ given the assumptions of Theorem 2. To show the existence of $M_1(h)$, we use the following argument:

$$\begin{aligned} |M_1| &\le \int_{-\infty}^{\infty} |xh(x)|\,dx \\ &\le \int_{-\infty}^{\infty} (1+|x^3|)h(x)\,dx \\ &\le M_0(h)+\rho(h). \end{aligned}$$

The result for M_2 follows similarly.

We now develop two lemmas.

LEMMA 1. *If*

$$\int_{-\infty}^{\infty} f(y)\,dy = 0 \quad \textit{and} \quad \int_{-\infty}^{\infty} |y^n f(y)|\,dy < \infty,$$

then

$$\lim_{y\to\pm\infty} \left| y^n \int_{-\infty}^{y} f(y_1)\,dy_1 \right| = 0.$$

Proof: We prove just the limit as $y \to \infty$. The opposite limit is proven in much the same manner. We can rewrite $|y^n \int_{-\infty}^{y} f(y_1)\,dy_1|$ as

$$\begin{aligned} \left| y^n \int_{-\infty}^{y} f(y_1)\,dy_1 \right| &= \left| y^n \left[\int_{-\infty}^{\infty} f(y_1)\,dy_1 - \int_{y}^{\infty} f(y_1)\,dy_1 \right] \right| \\ &= \left| y^n \int_{y}^{\infty} f(y_1)\,dy_1 \right|. \end{aligned}$$

If we restrict ourselves to $y > 0$, it follows that

$$\left| y^n \int_{-\infty}^{y} f(y_1)\,dy_1 \right| \le \int_{y}^{\infty} \left| y_1^n f(y_1) \right| dy_1,$$

where we have used the fact that $f \ge 0$. Taking the limit as $y \to \infty$, using the boundedness of $\int_{-\infty}^{\infty} |y_1^n f(y_1)|\,dy_1$, we get 0. ■

LEMMA 2. *Under the same assumptions as in Lemma 1*

$$\int_{-\infty}^{\infty} \left| y^{n-1} \int_{-\infty}^{y} f(y_1)\,dy_1 \right| dy \le \frac{1}{n} \int_{-\infty}^{\infty} |y^n f(y)|\,dy.$$

Proof: The integral can be expressed as $\int y^{n-1}(-1)^{\alpha} \int_{-\infty}^{y} f(y_1)\,dy_1$ where $\alpha = \pm 1$ depending on the sign of the integrand. Integrating by parts and noting that the derivative of $(-1)^\alpha |\int_{-\infty}^{y} f(y_1)\,dy_1|$ is $\tilde{f}(y)$ for some $\tilde{f}$ satisfying $|\tilde{f}(y)| = |f(y)|$, we get

$$\int_{-\infty}^{\infty}\left[y^{n-1}(-1)^\alpha \int_{-\infty}^{y} f(y_1)\,dy_1\right] dy$$
$$= \left|\frac{y^n}{n}\int_{-\infty}^{y} f(y_1)\,dy_1\right|_{y=-\infty}^{y=\infty} - \frac{1}{n}\int_{-\infty}^{\infty} y^n \tilde{f}(y)\,dy.$$

The first term vanishes by Lemma 1. Taking absolute values of the integrand in the second term finishes the proof. ∎

We are now able to prove Theorem 2.

Proof of Theorem 2: Note that $E(x,t)$ solves the heat equation, $E_t = E_{xx}$, for $-\infty < x < \infty$. The solution can be obtained by convoluting the heat kernel with the initial condition [8],

$$E(x,t) = \int_{-\infty}^{\infty} \frac{E(y,0)\,e^{-(x-y)^2/4t}}{\sqrt{4\pi t}}\,dy. \tag{10}$$

For notational simplicity, let $w_{yyy} = E(y,0)$ where $w(-\infty) = 0$. We can find $w(y)$ by integrating $y^2 E(y,0)$ by parts. We get

$$w(y) = \frac{y^2}{2}\int_{-\infty}^{y} E(y_1,0)\,dy_1 - y\int_{-\infty}^{y} y_1 E(y_1,0)\,dy_1 + \frac{1}{2}\int_{-\infty}^{y} y_1^2 E(y_1,0)\,dy_1.$$

The existence of $\rho(g)$ and $\rho(h)$ coupled with the integrability of g and h guarantees the integrability of $y^2E(y,0)$, $yE(y,0)$, and $E(y,0)$, so each term of w exists. We can similarly find w_y and w_{yy}.

We integrate (10) by parts three times. Because $E(y,0) = g(y) - h(y)$ and the zeroth, first, and second moments of h and g match, we can apply Lemma 1 each time to show that the contributions from w_y, w_{yy}, and w_{yyy} vanish yielding

$$E(x,t) = \frac{t^{-2}}{16\sqrt{\pi}}\int_{-\infty}^{\infty} H_3\left(\frac{x-y}{2\sqrt{t}}\right) e^{\frac{-(x-y)^2}{4t}}\, w(y)\,dy,$$

where H_3 is the third Hermite polynomial, $H_3(z) = 8z^3 - 12z$. Note that each successive integration by parts brings out a factor of $t^{-1/2}$.

This is a convolution integral, so using Young's inequality, it suffices to show that

$$\left\| H_3\left(\frac{x-y}{2\sqrt{t}}\right) e^{-(x-y)^2/4t}\right\|_p = \mathcal{O}(t^{1/2p}) \quad \text{and} \quad \|w\|_1 = \mathcal{O}(1)$$

to conclude $\|E(x,t)\|_p = \mathcal{O}(t^{-2+1/2p})$.

To obtain the first bound in Theorem 2, let $z = (x - y)/2\sqrt{t}$. We have that $dy = -2\sqrt{t}\,dz$. Evaluating the L_p norm gives

$$\begin{aligned}\left[\int_{-\infty}^{\infty}\left|H_3\left(\frac{x-y}{2\sqrt{t}}\right)e^{-(x-y)^2/4t}\right|^p dy\right]^{\frac{1}{p}} &= \left[\int_{-\infty}^{\infty}\left|H_3(z)e^{-z^2}\right|^p 2\sqrt{t}\,dz\right]^{\frac{1}{p}}\\ &= t^{1/2p}\left[2\int_{-\infty}^{\infty}\left|H_3(z)e^{-z^2}\right|^p dz\right]^{\frac{1}{p}}\\ &= \mathcal{O}(t^{1/2p}).\end{aligned}$$

To show $\|w\|_1 = \mathcal{O}(1)$, simply apply Lemma 2 term-by-term.

This gives bounds on $\|E\|_p$. We cannot improve this result unless the integral of w has finite L_1 norm. To ensure this we would require that a further moment of $E(y, 0)$ be zero.

To get the bound on $\|\int_{-\infty}^{y} E(s,t)\,ds\|_\infty$, we use the same technique as we used on E. We get an equation analogous to Equation (10):

$$\int_{-\infty}^{x} E(s,t)\,ds = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{y} E(s,0)\,ds\right]e^{-(x-y)^2/4t}\,dy.$$

In this case, only two moments go to zero, so we only integrate by parts twice. The L_∞ norm gains a factor of $t^{-1/2}$ with each integration, yielding a bound of order $t^{-3/2}$. ■

4. Return to Burgers' equation

Having bounded the error for the Gaussian asymptotic solution to the heat equation problem, we now return to Burgers' equation. Using the Cole–Hopf transformation (7) and Equation (9), we get

$$\begin{aligned}u(x,t) &= \frac{2\psi(x,t)}{\phi(x,t)}\\ &= \frac{2[G(x,t)+E(x,t)]}{\phi(x,t)}.\end{aligned}$$

We expect the solution corresponding to G to be an asymptotic approximation for u. We use (6) and (7) to give

$$\theta(x,t) = \frac{2G(x,t)}{1-\int_{-\infty}^{x} G(s,t)\,ds}$$

as the Burgers' equation solution corresponding to the Gaussian solution to the heat equation, G. Now consider the difference between u and θ,

$$
\begin{aligned}
u(x,t)-\theta(x,t) &= \frac{2\,[G+E]\left[1-\int_{-\infty}^{x} G\,ds\right]-2G\phi}{\phi(x,t)\left[1-\int_{-\infty}^{x} G\,ds\right]} \\
&= \frac{2G\int_{-\infty}^{x} E\,ds+2E\left[1-\int_{-\infty}^{x} G\,ds\right]}{\phi(x,t)\left[1-\int_{-\infty}^{x} G\,ds\right]},
\end{aligned} \tag{11}
$$

where in the second line we have substituted for ϕ in the numerator using Equation (6) and performed some algebra.

We are now ready to prove Theorem 1 which states that the L_p error between the true solution and our self-similar asymptotic approximation is $\mathcal{O}(t^{-2+1/2p})$.

Proof of Theorem 1: To prove the theorem, we must find the necessary bounds on Equation (11). We will bound the denominator from below by a constant and then show that each term in the numerator is $\mathcal{O}(t^{-2+1/2p})$.

We bound the denominator using inequality (4). This, combined with the fact that $1-\int_{-\infty}^{x} G\,ds > 1-\int_{-\infty}^{\infty} G\,ds = \phi(\infty,0)$, shows that the denominator is at least $e^{-M_0(f)}$.

We find the L_p bound on the numerator by finding it for each term. Using our bound $\|\int E\|_\infty = \mathcal{O}(t^{-3/2})$ from Theorem 2, the L_p norm of the first term of the numerator is

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty}\left|2G(x,t)\int_{-\infty}^{x} E(s,t)\,ds\right|^p dx\right)^{\frac{1}{p}} &\le \left(\int_{-\infty}^{\infty}\left|\frac{Ke^{-(x-x_*)^2/4(t+t_*)}}{t^{3/2}\sqrt{4\pi(t+t_*)}}\right|^p dx\right)^{\frac{1}{p}} \\
&\le \frac{K}{t^{3/2}\sqrt{4\pi(t+t_*)}}\left(\int_{-\infty}^{\infty} e^{\frac{-p(x-x_*)^2}{4(t+t_*)}}dx\right)^{\frac{1}{p}}
\end{aligned}
$$

for some constant K. Letting $\zeta=(x-x_*)/2\sqrt{t+t_*}$ and integrating, we get that this is $\mathcal{O}(t^{-2+1/2p})$.

To bound the second term of the numerator, we use the fact that $\|E\|_p = \mathcal{O}(t^{-2+1/2p})$ from Theorem 2 and the fact that $\phi(\infty,0) < 1-\int_{-\infty}^{x} G(s,t)\,ds < 1$.

Thus the two terms have the desired L_p norm, and so their sum does as well. This finishes the proof. ■

5. An example

The improvement given by the shifts in x and t is illustrated in Figure 1 for the tophat initial condition,

$$
f(x)=\begin{cases} 0 & x\le 0 \\ 4 & 0<x<2 \\ 0 & 2\le x. \end{cases} \tag{12}
$$

The analytic solution is

$$u = \frac{-4e^{4t-2x}\left[\operatorname{erf}\left(\frac{2-x+4t}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{-x+4t}{2\sqrt{t}}\right)\right]}{-\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) - e^{-4}\operatorname{erfc}\left(\frac{2-x}{2\sqrt{t}}\right) + e^{4t-2x}\left[\operatorname{erf}\left(\frac{-x+4t}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{2-x+4t}{2\sqrt{t}}\right)\right]}.$$

We transform to the heat equation to get x_* and t_*. The initial conditions transform as

$$\psi(x,0) = \begin{cases} 0 & x \le 0 \\ 2e^{-2x} & 0 < x < 2 \\ 0 & 2 \le x. \end{cases}$$

We compute the moments of ψ to be

$$M_0(\psi) = 1 - e^{-4}$$
$$M_1(\psi) = \frac{1 - 5e^{-4}}{2}$$
$$M_2(\psi) = \frac{1 - 13e^{-4}}{2},$$

from which we get

$$C = 1 - e^{-4}$$
$$x_* = \frac{1 - 5e^{-4}}{2 - 2e^{-4}}$$
$$t_* = \frac{1 - 18e^{-4} + e^{-8}}{8(1 - e^{-4})^2}.$$

Then Equation (8) yields the asymptotic self-similar approximation for the heat equation,

$$G(x,t) = C\frac{\exp\left[-(x - x_*)^2/4(t + t_*)\right]}{\sqrt{4\pi(t + t_*)}}.$$

Returning to Burgers' equation via the Cole–Hopf transformation yields our asymptotic solution,

$$\theta(x,t) = \frac{2G(x,t)}{1 - \int_{-\infty}^{s} G(s,t)\,ds},$$

which is plotted in Figure 1. Chern and Liu's approximation is given by setting $x_* = 0$ and $t_* = 1$.

6. A remark on asymptotics

An asymptotic expansion for the solution in terms of powers of $1/\sqrt{t}$ can be derived from the Fourier transform solution of the heat equation. Suppose $\psi(x,t)$

satisfies the heat equation $\psi_t = \psi_{xx}$ on the infinite line with initial condition $\psi(x, 0) = h(x)$. We can solve this problem using the Fourier transform; define $\widehat{h}(k) = \int_{-\infty}^{\infty} h(x)e^{-ikx}\,dx$ and assume that all the moments $M_n(h)$ are bounded.

The Taylor series of $\widehat{h}(k)$ can be expressed as

$$\widehat{h}(k) = \sum_{j=0}^{\infty} \frac{\widehat{h}^{(j)}(0)}{j!} k^j.$$

Using the definition of the Fourier transform

$$\widehat{h}(k) = \int_{-\infty}^{\infty} h(x)\,e^{-ikx}\,dx = \sum_{j=0}^{\infty} (-i)^j k^j \int_{-\infty}^{\infty} h(x)x^j\,dx = \sum_{j=0}^{\infty} \frac{(-i)^j M_j(h)}{j!} k^j,$$

and equating the two expressions for $\widehat{h}(k)$ yields the moment identity, $\widehat{h}^{(j)}(0) = (-i)^j M_j(h)$.

We can now develop an asymptotic expansion for ψ by writing its solution in terms of an inverse Fourier transform, replacing $\widehat{h}$ by its formal Taylor series, and exchanging summation and integration,

$$\begin{aligned}
\psi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{h} e^{-k^2 t} e^{ikx}\,dk \\
&\sim \sum_{j=0}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(-i)^j M_j(h)}{j!} k^j e^{-k^2 t} e^{ikx}\,dk \\
&\sim \sum_{j=0}^{\infty} \frac{M_j(h)(-1)^j}{j!} \frac{d^j}{dx^j} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 t} e^{ikx}\,dk \right) \\
&\sim \sum_{j=0}^{\infty} \frac{M_j(h)(-1)^j}{j!} \frac{d^j}{dx^j} \left(\frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \right) \\
&\sim \frac{1}{\sqrt{4\pi t}} \sum_{j=0}^{\infty} \frac{M_j(h)(-1)^j}{j!} \left(\frac{1}{2\sqrt{t}} \right)^j e^{-x^2/4t} H_j\left(\frac{x}{2\sqrt{t}} \right),
\end{aligned}$$

where H_j is the jth Hermite polynomial. An asymptotic expansion for u can be recovered by substituting into Equations (6) and (7) and expanding in powers of $1/\sqrt{t}$.

The approximation of Chern and Liu corresponds to retaining only the first term in this series; the second term, which is $\mathcal{O}(t^{-1})$, yields the error estimate. Our approximation corresponds to first making the change of variables $x \mapsto x - x_*$ and $t \mapsto t + t_*$, which causes M_1 and M_2 to vanish in this expansion. The first non-zero term, which is now $\mathcal{O}(t^{-2})$, is responsible for our error estimate.

Providing general conditions under which the sum converges is beyond the scope of this paper. However, if $h(x) \leq A \exp[-c|x|^s]$ with $A, c > 0$ and $s > 2$

then the series converges for all $t > 0$. Note that this guarantees convergence for the example of the previous section. Moreover, as long as the $\mathcal{O}(t^{-2})$ error term does not vanish in our expansion of ψ, the error bound in Theorem 1 will be sharp.

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References

1. G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, New York, 1974.
2. R. E. Grundy, Asymptotic solution of a model nonlinear convection diffusion equation, *IMA J. Appl. Math*. 31:121–137 (1983).
3. I.-L. Chern, Long-time effect of relaxation for hyperbolic conservation laws, *Commun. Math. Phys*. 172:39–55 (1995).
4. I.-L. Chern, Multiple-mode diffusion waves for viscous nonstrictly hyperbolic conservation laws, *Commun. Math. Phys*. 138:51–61 (1991).
5. I.-L. Chern and T.-P. Liu, Convergence to diffusion waves of solutions for viscous conservation laws, *Commun. Math. Phys*. 110:503–517 (1987).
6. M. Escobedo and E. Zuazua, Long-time behavior for a convection-diffusion equation in higher dimensions, *SIAM J. Math. Anal*. 28:570–594 (1997).
7. E. Zuazua, A dynamical system approach to the self-similar large time behavior in scalar convection-diffusion equations, *J. Differ. Equ*. 108:1–35 (1994).
8. L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.
9. G. Kleinstein and L. Ting, Optimum one-term solutions for the heat conduction problems, *Z. Agnew. Math. Mech*. 51:1–16 (1971).
10. T. P. Witelski and A. J. Bernoff, Self-similar asymptotics for linear and nonlinear diffusion equations, *Stud. Appl. Math*. 100:153–193 (1998).

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